March 30: Krull Domains and the Mori-Nagata Theorem, part 1

Krull Domains

The purpose of this part of the course is to address the degree to which the integral closure of a Noetherian domain fails to be Noetherian. In the previous section, we saw that Nagata's example shows that the integral closure of a one-dimensional Noetherian domain R need not be a finite R-module. It is, however, a Noetherian ring. This will follow from the results below.

As mentioned above, the integral closure of a two-dimensional Noetherian domain is again Noetherian, but this fails for Noetherian domains of dimension greater than two. This failure is mitigated by the fact that the integral closure is Noetherian-like in codimension one.

This is made precise by saying that the integral closure of a Noetherian domain is a Krull domain, a fact known as the Mori-Nagata theorem.

Therefore, the purpose of this part of the course is to prove the Mori-Nagata theorem.

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We begin with a definition.

Definition. Let S be an integral domain with quotient field L. We say that S is a Krull domain if the following conditions hold.

(i) Each nonzero element of S is contained in only finitely many height one primes.

(ii) S_Q is a DVR, for all height one primes $Q \subseteq S$.

(iii) $S = \bigcap_{\text{height}(Q)=1} S_Q$.

There are a number of ways that a Krull domain behaves like an integrally closed Noetherian domain in codimension one.

We illustrate a few of these ways in the proposition below.

Proposition A2. The following properties hold.

(a) A Krull domain is integrally closed.

(b) An integrally closed Noetherian domain is a Krull domain.

(c) A Krull domain satisfies the ascending chain condition on principal ideals.

(d) If S is a Krull domain, $0 \neq a \in S$, and $Q_1, \ldots, Q_r$ are the height one prime ideals containing aS, then there exist $e_1, \ldots, e_r \geq 1$ such that $aS = Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$ is an irredundant primary decomposition of the principal ideal aS.

(e) If S is a Krull domain and $Q \subseteq S$ is a height one prime, then for any non-zero $a \in Q$, there exists $b \in S$ with $Q = (aS : b)$.

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Proof. It is easy to see that an intersection of integrally closed integral domains is integrally closed. Thus (a) follows from (ii) and (iii) in the definition of Krull domain.

For (b), Let R be an integrally closed Noetherian domain. Condition (i) holds in R since R is Noetherain, and the height one primes containing a principal ideal aR must be minimal over aR.

Condition (ii) holds by Corollary B, since a prime minimal over an ideal is an associated prime of the ideal.

Condition (iii) follows from the fact that $R = \bigcap_{P \in \mathcal{P}(R)} R_P$, where $\mathcal{P}(R)$ is the set of prime ideals associated to a non-zero principal ideal, and in case R is integrally closed, $\mathcal{P}(R)$ is just the set of height one prime ideals.

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For part (c), let $a_1 S \subseteq a_2 S \subseteq \cdots$ be an ascending chain of principal ideals. Let Q be a height one prime not containing $a_1 S$. Then $a_1 S_Q = S_Q$, and thus $a_n S_Q = S_Q$ for all n. Hence $a_1 S_Q = a_n S_Q$ for all n.

Now let X be the finite set of height one primes containing $a_1 S$. Take $Q \in X$. Then since S_Q is a DVR, there exists r, depending on Q, such that $a_r S_Q = a_n S_Q$, for all $n \geq r$. Since there are only finitely many primes in X, we can take n_0 the maximum of the r's we just found.

It follows that $a_{n_0} S_Q = a_n S_Q$, for all $n \geq n_0$ and all height one primes $Q \subseteq S$. This means $\frac{a_n}{a_{n_0}} \in S_Q$ for all height one primes Q, and so by property (iii) in the definition of Krull domain $\frac{a_n}{a_{n_0}} \in S$, for all $n \geq n_0$.

Thus $a_n \in a_{n_0} S$ for all $n \geq n_0$ and therefore the given ascending chain stabilizes at n_0.

For part (d), first recall that if Q is a prime ideal in a commutative ring A, then we define the n^{th} symbolic power of Q to be the ideal $Q^nA_Q \cap A$. Since Q^nA_Q is Q_Q-primary, $Q^{(n)}$ is Q-primary.

Now fix a non-zero element $a \in S$ and let $Q_1, \dots, Q_r$ be the height one prime ideals containing a. Let π_i be the uniformizing parameter for the DVR S_{Q_i} i.e., $\pi_i S_{Q_i} = Q_i S_{Q_i}$, for all i. Then, there exist $e_1, \dots, e_r$ such that $aS_{Q_i} = \pi_i^{e_i} S_{Q_i} = Q_i^{e_i} S_{Q_i}$, for all i. Thus, $aS \subseteq Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$.

Now let $x \in Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$. Then $x \in aS_{Q_i}$ for all i. Let Q be a height one prime not containing a.

Then $aS_Q = S_Q$, and hence $x \in aS_Q$. Thus $x \in aS_Q$, for all height one primes Q in S. In other words, $\frac{x}{a} \in \bigcap_{\mathrm{height}(Q)=1} S_Q = S$.

Thus, $x \in aS$, which shows $Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)} \subseteq aS$, which is what we want.

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Finally, the intersection is irredundant, since the nilradicals of the Q_i are distinct.

So for instance, if $aS = Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)}$, then $Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)} \subseteq Q_1$. But then some $Q_i^{(e_i)} \subseteq Q_1$ which implies $Q_i \subseteq Q_1$, a contradiction.

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For part (e), Let $Q \subseteq S$ be a height one prime and $0 \neq a \in Q$. Take a primary decomposition of aS as in part (d), and assume $Q = Q_1$.

By prime avoidance, we can find $b^* \in Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)} \backslash Q$. Take $b_0 \in Q^{(e-1)} \backslash Q^{(e)}$ and set $b := b^* b_0$.

If $c \in Q$, then $cb_0 \in Q^{(e)}$, and thus $cb \in aS$. On the other hand, if $cb \in aS \subseteq Q^{(e)}$, then $cb_0 \in Q^{(e)}$, by the choice of b^*.

Thus, $cb_0 \in \pi^e S_Q$, where $\pi S_Q = QS_Q$.

Since $b_0 \in \pi^{e-1} S_Q$, we have $c \in \pi S_Q \cap S = Q$, which is what we want.

Thus, $Q = (aS : b)$. □

Remark. Maintain the notation from part (d) in the Proposition above. Then for the given $a \in S$ as in (d), for all $n \geq 1$, $a^n S_{Q_i} = \pi_i^{ne_i} S_{Q_i}$.

Thus, arguing as before, it follows that $a^n S = Q_1^{(ne_1)} \cap \cdots \cap Q_r^{(ne_r)}$ is an irredundant primary decomposition of $a^n R$, for all $n \geq 1$. Here we are using the fact that aS and $a^n S$ are contained in exactly the same set of height one prime ideals.

Examples. (a) Any UFD is easily seen to be a Krull domain. Thus, for example, if K is a field, the polynomial ring in countably many variables over K is a non-Noetherian UFD, and hence a non-Noetherian Krull domain.

(b) If R is a Krull domain, then a polynomial ring in countable many variables over R is a Krull domain. Thus if $R = K[x, y, z, w]/(xy - zw)$, then adjoining countably many variables yields a non-Noetherian Krull domain that is not a UFD.

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The following technical proposition due to J. Nishimura has a number of applications, including the lovely theorem which follows it.

Proposition B2. Let S be a Krull domain and $Q \subseteq S$ a height one prime ideal. Then, for all $n \geq 1$, the S-module $Q^{(n)}/Q^{(n+1)}$ embeds into S/Q.

Proof. Take $0 \neq a \in Q$. We can write $Q = (aS : b)$, for some b. If we take a primary decomposition

$$aS = Q^{(e)} \cap Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)},$$

the proof of part (d) of the previous proposition shows that we can assume $b \in Q^{(e-1)} \cap Q_2^{(e_2)} \cap \cdots \cap Q_r^{(e_r)}$. We claim that if $x \in Q^{(n)}$, then $x \in (a^n S : b^n)$. Thus, $x \cdot \frac{b^n}{a^n} \in S$.

To see the claim, take $s \in S \backslash Q$ such that $sx \in Q^n$. Then $sxb^n \in a^n S$. Since $Q^{(en)}$ is the Q-primary component of $a^n S$ and $s \notin Q$, $xb^n \in Q^{(en)}$.

On the other hand, $b^n \in Q_2^{(ne_2)} \cap \cdots \cap Q_r^{(ne_r)}$, so $xb^n \in a^n S$.

We thus have an S-module map $Q^{(n)} \xrightarrow{\frac{b^n}{a^n}} S \to S/Q$. Call this map ϕ. We need to show that if $x \in Q^{(n)}$, then $\phi(x) \in Q$ if and only if $x \in Q^{(n+1)}$. If so, then ϕ induces an injective map from $Q^{(n)}/Q^{(n+1)}$ into S/Q, as required.

Take $x \in Q^{(n)}$ and assume $\pi \in Q$ satisfies $\pi S_Q = QS_Q$. Suppose $\phi(x) \in Q$. Then $x \cdot \frac{b^n}{a^n} \in Q$, so $xb^n \in Qa^n$.

Therefore, $xb^n \in \pi a^n S_Q$. But in S_Q, $b = u\pi^{e-1}$ and $a \in \pi^e S_Q$, where $u \in S_Q$ is a unit. Thus $x\pi^{n(e-1)} \in \pi^{en+1} S_Q$.

It follows that $x \in \pi^{n+1} S_Q \cap S = Q^{(n+1)}$.

Conversely, suppose $x \in Q^{(n+1)}$, Then there exists $s \in S \backslash Q$ such that $sx \in Q^{n+1}$. Then $sxb^n \in a^n Q$ (since $Q = (aS : b)$).

Therefore, $s(x \cdot \frac{b^n}{a^n}) \in Q$. In other words, $s \cdot \phi(x) \in Q$. But $\phi(x) \in S$ and $s \notin Q$, so $\phi(x) \in Q$, which is what we want. □

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Theorem C2. (Nishimura) Let S be a Krull domain. If S/Q is Noetherian for all height one primes $Q \subseteq S$, then S is Noetherian.

Proof. Let $I \subseteq S$ be an ideal and take a non-zero $a \in I$. It suffices to show that I/aS is a finitely generated S-module.

Take the primary composition $aS = Q_1^{(e_1)} \cap \cdots \cap Q_r^{(e_r)}$ as above, where the Q_i are the height one primes containing a. Then, on the one hand,

$$S/aS \hookrightarrow S/Q_1^{(e_1)} \oplus \cdots \oplus S/Q_r^{(e_r)},$$

so it suffices to show that $S/Q_1^{(e_1)} \oplus \cdots \oplus S/Q_r^{(e_r)}$ is a Noetherian S-module.

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On the other hand, given any height one prime $Q \subseteq S$, our assumption on Q and Proposition B2 show that $Q^{(n-1)}/Q^{(n)}$ is a Noetherian S/Q-module, and hence a Noetherian S-module, for all $n \geq 1$.

Thus, induction on n and the short exact sequences

$$0 \to Q^{(n-1)}/Q^{(n)} \to S/Q^{(n)} \to S/Q^{(n-1)} \to 0$$

show that each $S/Q^{(n)}$ is a Noetherian S-module.

Therefore, $S/Q_1^{(e_1)} \oplus \cdots \oplus S/Q_r^{(e_r)}$ is a Noetherian S-module, which completes the proof. □